Problem Set 3

Advanced Statistical Mechanics

Deadline: Friday, 3 Ordibehesht, 23:59

1 Stochastic Processes

1.1 Rabbit Evolution

Consider an experiment of mating rabbits. We watch the evolution of a particular gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid-the order is irrelevant, so gG is the same as Gg) or gg (recessive). In mating two rabbits, the offspring inherits a genre from each of its parents with equal probability. Thus, if we mate a dominant (GG) with a hybrid (Gg), the offspring is dominant with probability 1/2 or hybrid with probability 1/2.

Start with a rabbit of given character (GG, Gg, or gg) and mate it with a hybrid. The offspring produced is again mated with a hybrid, and the process is repeated through a number of generations, always mating with a hybrid.
(i) Write down the transition probabilities of the Markov chain thus defined.
(ii) Assume that we start with a hybrid rabbit. Let μ_n be the probability distribution of the character of the rabbit of the n-th generation rabbit is GG, Gg, or gg, respectively. Compute μ_1, μ_2, μ_3. Can you do the same for μ_n for general n?

1.2 Knight's Tour

Consider the knight's tour on a chess board: A knight selects one of the next positions at random independently of the past.
(i) Why is this process a Markov chain?
(ii) What is the state space?
(iii) Is it irreducible? Is it aperiodic?
(iv) Find the stationary distribution. Give an interpretation of it: what does it mean, physically?
(v) Which are the most likely states in steady-state? Which are the least likely ones?

1.3 Colored balls

Five with balls and five black balls are distributed in two urns is such a way that each urn contain five balls. At each step we draw one ball from each urn and exchange them.

Let X_n be the number of white balls in the left urn at time n.
(i) Compute the transition probability for X_n
(ii) Find the stationary distribution and show that it corresponds to picking five balls at random to be in the left urn.

1.4 Transition Matrix

(i) Consider a homogeneous Markov chain with transition matrix

$$\begin{pmatrix} q_1 & p_1 & 0 & 0 & 0 & \cdots \\ q_2 & 0 & p_2 & 0 & 0 & \cdots \\ q_3 & 0 & 0 & p_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

where $p_i = 1 - q_i$ and $p_i, q_i \geq 0$, $i \in \mathbb{N}$. The chain is irreducible if $0 < p_i < 1$ for all $i \in \mathbb{N}$. Find the necessary and sufficient conditions for transience positive recurrence, null recurrence respectively.
(ii) Consider a homogeneous Markov chain with state space $S = \{1, \cdots, n\}$ and transition matrix

$$\begin{pmatrix} q & p & 0 & 0 & 0 & \cdots & 0 \\ q & 0 & p & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & p & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & q & 0 & p \\ 0 & \cdots & 0 & 0 & 0 & q & p \end{pmatrix} \tag{2}$$

Show that the chain is positive and find the stationary initial distribution.

1.5 Random Walk

Consider an asymmetric random walk on an open-ended lattice with four lattice sites. The transition rates are $w_{1,2} = w_{4,3} = 1$, $w_{2,3} = w_{3,4} = \frac{3}{4}$, $w_{2,1} = w_{3,2} = \frac{1}{4}$, and $w_{i,j} = 0$ for all other transitions.
(i) Write the transition matrix, $W_{m,n}$, and show that the system obeys detailed balance.
(ii) Compute the matrix $V_{m,n}$ and find its eigenvalues and eigenvectors.
(iii) Write $P_1(n,t) = \delta_{n,1}$. What is $P_1(2,t)$

1.6 Birth and Death Processes

An important class of master equations respond to the birth and death scheme. Let us assume that particles of a system can be in the state X or Y. For instance, we could think of a person who is either sane or ill. The rates of going from X to Y is ω_1 while the rate from Y to X is ω_2. Recall that both rates ω_1 and ω_2 are independent of each other.

Let n_1 be the number of particles in the state X while n_2 is the number of particles in the state Y. The total number of particles is $N = n_1 + n_2$, constant.

(i) Write a master equation for the probability $P(n;t)$ of having n particles in state X.
(ii) Now we need to solve this master equation. So by using the generating function $G(s,t)$, rewrite the equation.
(iii) If initially there are no "life" particles, then calculate $G(s,t)$
(iv) Calculate the evolution of the average number of particles and their variance.

1.7 LGKS Equation

In general the dynamics of the reduced system defined by the exact equations

$$\rho_S(t) = \mathrm{tr}_B(U(t,t_0)\rho(t_0)U^\dagger(t,t_0) \tag{3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -\frac{i}{\hbar}\,\mathrm{tr}_B[H(t),\rho(t)] \tag{4}$$

will be quite involved. However, under the condition of short environmental correlation times one may neglect memory effects and formulate the reduced system dynamics in terms of a quantum dynamics semigroup.

Let us suppose that we are able to prepare at the initial time $t = 0$ the state of the total system $\mathbf{S} + \mathbf{B}$ as an uncorrelated product state $\rho(0) = \rho_S(0) \otimes \rho_B$, where ρ_S is the initial state of the reduced system $\mathbf{S}$ and ρ_B represents some reference state of the environment, a thermal equilibrium, for example. The transformation describing the change of the reduced system from the initial time $t = 0$ for some $t > 0$ may then be written in the form

$$\rho_S(0) \to \rho_S(t)V(t)\rho_S(0) \equiv \mathrm{tr}_B(U(t,0)[\rho_S(0),\rho_B]U^\dagger(t,0)) \tag{5}$$

If we regard the reference state ρ_B and the final time to be fixed, this relation defines a map from the space $D(\mathcal{H}_S)$ of density operator of the reduced system into self,

$$V(t) : D(\mathcal{H}_S) \to D(\mathcal{H}_S) \tag{6}$$

This map, describing the state change of the open system over time t, is called a dynamical map. If the characteristic time scales over which the reservoir correlation functions decay are much smaller than the characteristic time scale of the systematic system evolution, it is justified to neglect memory effects in the reduced system dynamics. As in the classical theory one thus expects Markovian-type behavior. For the homogeneous case the latter may be formalized with the help of the semigroup property:

$$V(t_1)V(t_2) = V(t_1 + t_2), \quad t_1, t_2 > 0 \tag{7}$$

Given a quantum dynamical semigroup there exists, under certain mathematical conditions, a linear map $\mathcal{L}$, the generator of the semigroup, which allows us to represent the semigroup in exponential form,

$$V(t) = \exp(\mathcal{L}t) \tag{8}$$

This representation immediately yields a first-order differential equation for the reduced density operator for the open system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_s(t) = \mathcal{L}\rho_S(t) \tag{9}$$

which is called the Markovian quantum master equation.

Show that the most general form of the generator $\mathcal{L}$ of a quantum dynamical semigroup is

$$\mathcal{L}\rho_S = -\frac{i}{\hbar}[H, \rho_S] + \sum_k \gamma_k \left(A_k \rho_S A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho_S\} \right) \tag{10}$$

which is called LGKS equation.